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On the effects of boundary-layer growth on flow stability

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The stability of small travelling-wave disturbances in the flow over a flat plate is discussed. An iterative method is used to generate an asymptotic series solution in inverse powers of the Reynolds number $R_x = Ux/\nu$ to the power one half. The neutral-stability boundaries given by the first two terms of this series are obtained and compared with experimental data. It is shown that the parallel flow approximation leads to a valid solution at very large Reynolds numbers.

1. Introduction

The prediction of the stability of a given flow and the subsequent amplification of any small disturbance has been of continuing interest to the fluid dynamicist for nearly a century. The initial stages of development of any disturbance are described by a set of perturbation equations derived from the Navier–Stokes equations by linearization. Any general perturbation can be split into normal modes and the flow is deemed to be stable or unstable depending on the behaviour of the least stable mode. Thus, in general, the determination of the stability of a given flow requires the evaluation of the eigenvalues of a set of partial differential equations. In some cases, where the mean flow field is defined as a function of only one spatial variable, the set of partial differential equations defining the perturbation is separable, and the problem reduces to that of solving a set of ordinary differential equations. Plane Poiseuille flow is such an example. In many practical cases, where the mean flows do not *exactly* fall within the above category, the dependence on a second variable is weak and the approximation is commonly made that the stability characteristics are related in some sense to those of the equivalent one-dimensional system. Boundary layers are examples of this type: the mean flows are almost one-dimensional and their stability characteristics are invariably determined by making the approximation that the flow *is* parallel over some region. The relevant perturbation equations then separate and the ordinary differential equation obtained is known as the Orr–Sommerfeld equation. This equation applies strictly to disturbances in real parallel flows, but it has also been used to describe the wave systems in other flows such as jets and wakes, which are nearly parallel. In the case of the flat-plate boundary layer this approach appears to be a reasonable one, since the terms neglected are of order $R_x^{-\frac{1}{2}}$ smaller than those retained (Pretsch 1941), but there has been no formal justification for using this approximation.

The difficulty in correctly formulating the stability problem for slightly non-parallel flows has led to most of the significant theoretical effort being directed towards genuine parallel flows. It is, however, much simpler to experiment on boundary layers, for which, not surprisingly, most detailed experimental data exist.

The instability of boundary-layer flow was theoretically demonstrated by Tollmien (1929) and Schlichting (1933) using the parallel mean flow approximation. They predicted a critical value of the boundary-layer Reynolds number above which travelling-wave disturbances grew. These results were derived from rather approximate asymptotic solutions of the Orr–Sommerfeld equation, valid only at high Reynolds number. Partly because of the number and the nature of the approximations used to reach the final results, and partly because there was at that time no supporting experimental evidence, these results were received with a certain degree of scepticism.

Schubauer & Skramstad (1948) carried out a remarkable series of experiments on the flat-plate boundary layer to study these waves. Having reduced the free-stream turbulence in the wind tunnel to a very low level they were able to detect irregular travelling waves in the boundary layer within the predicted frequency band. They also showed that these disturbances amplified in the manner predicted by theory. The use of a wave maker, in the form of a vibrating ribbon, allowed detailed measurements on these travelling waves to be made. These artificially excited regular waves exhibited characteristics remarkably close to those predicted by the theory, and the agreement with the quasi-parallel flow treatment was sufficiently good to establish the theory as capable of describing the unstable waves which arise in the initial stages of transition.

For a parallel mean flow defined by $U(y)$ the appropriate perturbation stream function is

$$\psi(x,y,t) = \phi(y)\exp i(\alpha x-\beta t),$$

where α, the wavenumber, is generally taken to be real and $\beta = \beta_r + i\beta_i$, the frequency parameter, is complex. A positive value of β_i is taken to denote instability to the perturbation. Schlichting used the parallel mean flow approximation for the flat-plate boundary layer and obtained the eigenvalues α and β and the eigenfunction $\phi(y)$ for a range of Reynolds numbers R. The Reynolds number occurs in the Orr–Sommerfeld equation as a parameter. From the information on the local growth rate, he suggested that the overall amplification of a disturbance of a fixed frequency was given by the integral

$$\int_{x_0}^{x_1} \frac{\beta_i(x)}{\partial\beta_r(x)/\partial\alpha}\,dx. \tag{1}$$

The factor $\partial\beta_r/\partial\alpha$ appearing in (1) was used to relate the spatial growth in the physical situation to the temporal growth employed in the theory. In fact, a better description of the behaviour of regular periodic waves is given by the spatial modes, for which α is treated as complex and β as real (see Gaster 1965). The growth of these waves is given by

$$-\int_{x_0}^{x_1} \alpha_i(x)\,dx. \tag{2}$$

In the case of weakly amplified or damped waves it has been shown that (1) and (2) are equivalent (Gaster 1962). Schubauer & Skramstad chose to relate the measured growth with theory by obtaining the amplification from

$$\int \frac{\beta_i(x)}{\beta_r/\alpha}\,dx, \tag{3}$$

where β_r/α is the phase velocity. The phase velocity differs from the group velocity by a maximum of about 20 % for flat-plate boundary-layer modes. The differences between the relations (1), (2) and (3) are less significant than errors introduced by using eigenvalues obtained from the asymptotic solutions of the Orr–Sommerfeld equation.

The general concept of an instability and the physics of the processes involved in its evolution can well be discussed within the framework of the parallel flow approximation, but detailed comparison with experiment cannot be made in a meaningful way to the necessary degree of precision. Solutions of the Orr–Sommerfeld equations can now be found to considerable accuracy by direct numerical integration. The more critical comparison with experiment which is now possible demands a clearer statement as to precisely how solutions for the locally parallel flow can be incorporated into a *consistent* theory describing the behaviour of waves over large regions of a real growing boundary layer.

In some recent work on wave packets (Gaster & Grant 1974) it was necessary to obtain a theoretical estimate of the phase as well as the amplitude for each travelling wave mode. The stream function chosen to define these waves was of the form

$$\psi = A(x)\,\phi(y,x)\exp i\{\textstyle\int \alpha(x)\,dx - \beta t\}, \tag{4}$$

where α and ϕ were evaluated locally from the Orr–Sommerfeld equation as functions of x. It was felt that the factor A should also be incorporated in the description of these waves to take some account of the variations of the wave-number and eigenfunction with x. In cases where such variations are extremely slow (large R), it is perhaps reasonable to neglect any such weak algebraic term and accept the limiting form with constant A. Some additional type of conservation relation, such as wave action, enables $A(x)$ to be determined. Such techniques are commonly used in reducing a partial differential equation which has a weak dependence on one variable to an ordinary one. Integral methods used in calculating steady laminar boundary layers are often of this type: the 'local' solutions give good estimates of the shape of the velocity profile, but quantities like the momentum thickness which are controlled by the history of the boundary-layer development require an additional scaling parameter obtained from conservation relations like the momentum integral equation.

Here an attempt is made to generate a sequence of terms for the stream function of a periodic travelling-wave disturbance in a growing boundary layer. A direct perturbation expansion in some small parameter characterizing the mean flow divergence seems to be the most direct approach. For the boundary-layer problem an appropriate small parameter is $R^{-\frac{1}{2}}$. There are, however, two length scales in the problem related to this parameter: in addition to the co-ordinate stretching scale arising from the boundary-layer development, there is also the length scale associated with the viscous inner and critical layers of the Orr–Sommerfeld solution. Bouthier (1972, 1973) has applied the method of multiple scales to the boundary-layer problem by artificially separating out the small parameter controlling the distortion of the co-ordinates from the viscous term in the equations of motion. This procedure may be dangerous since no account of the vertical structure of the Orr–Sommerfeld solutions is taken in ordering the terms in the expansion. In regions where the viscous terms dominate, differentiation in the normal direction raises the order of a term by $R_x^{\frac{1}{2}}$, and multiple operations may make it necessary to consider additional terms in the expansion. A similar expansion scheme which neglects the vertical structure has also been used by Ling & Reynolds (1973) to obtain estimates of changes in wavelength and amplification rate for a number of slowly varying flows, including the boundary layer on a flat plate.

Unless the two small parameters which arise in the above treatment are separated artificially the expansion process has to be carried out separately in the various layers. These matched expansions presumably lead to an asymptotic series solution, the leading term of which is the asymptotic solution of the Orr–Sommerfeld equation. Since these basic solutions are poor at the moderate Reynolds number associated with boundary-layer instabilities, this method does not seem promising. A more direct approach, which is used here, is to employ an iteration scheme to develop the series. The parallel flow approximation yields a suitable trial solution and successive correction terms lead to a series in descending powers of $R^{\frac{1}{2}}$.

2. Analysis

Some initial disturbance is assumed to exist at a station x_0 and solutions are sought for $x > x_0$, where $x/x_0 - 1$ is not necessarily small. It is convenient to choose new co-ordinates (ξ, η) to rescale the problem so that the domain over which the solution is sought is a rectangle with sides of order unity.

$$\xi = \frac{x}{x_0}, \quad \eta = y\left(\frac{U}{\nu x}\right)^{\frac{1}{2}} = \frac{U}{\nu}\frac{y}{R^{\frac{1}{2}}\xi^{\frac{1}{2}}}, \tag{5}$$

where U is the free-stream velocity, ν the kinematic viscosity and $R = Ux_0/\nu$ the Reynolds number at x_0. At this stage it is convenient to assume that the mean flow stream function $\overline{\psi}$ is known and that it is a smooth function of ξ and η throughout the region where the behaviour of the disturbance is to be evaluated.

The disturbed flow, like the undisturbed steady flow, must obey the appropriate governing equations: the Navier–Stokes equations. Since the superimposed disturbance is assumed to be small these equations may be linearized with respect to the perturbation ψ, and after subtracting out the mean flow terms the partial differential equation $L_1[\psi] = 0$ is formed. $L_1[\;]$, the linearized Navier–Stokes operator appropriate to the mean flow $\overline{\psi}$, defines the behaviour of the perturbation. [This operator is defined in appendix A, equation (A 1).]

Partial differential equations are generally only amenable to analytic solution if a co-ordinate system can be found which enables the operator to be separated.

The operator $L_1[\ ]$ does not separate unless certain terms are ignored. These terms are of order $R_x^{-\frac{1}{2}}$ and are suitably small in problems concerned with boundary-layer instability (Pretsch 1941). An approximate solution can thus be found with the wavenumber α, as the separation parameter, remaining a weak function of ξ. Such a solution is

$$\psi_0 = A(\xi)\,\phi_0(\xi,\eta)\,e^Q, \tag{6}$$

with

$$Q = i\left\{R^{\frac{1}{2}}\int_1^{\xi}\frac{\alpha(\xi)}{\xi^{\frac{1}{2}}}\,d\xi - \omega t\right\},$$

where ϕ_0 satisfies the Orr–Sommerfeld equation $L_2[\phi_0] = 0$, which is defined in appendix A. It is convenient to incorporate the term $\xi^{-\frac{1}{2}}$ in the spatial scale so that the parameter $\alpha(\xi)$ defines the wavelength in terms of the local boundary-layer thickness. ϕ_0 is normalized in some way convenient for the calculation scheme and $A(\xi)$ provides the necessary scaling. A is an arbitrary weak function of ξ at this level of approximation.

The above approximation to the disturbance stream function can be used as a trial solution in an iterative scheme to generate a series solution. The approximation which was used to obtain this trial solution is essentially that of replacing the operator $L_1[e^Q\ ]$, which does not separate, by the form $e^Q L_2[\ ]$, which does. The terms neglected are $O(R_x^{-\frac{1}{2}})$, but it does not seem necessary at this stage to justify this statement fully, although this can be accomplished by combining (6) with (A 1) and (A 2), and comparing the result with the Orr–Sommerfeld equation. The test of whether or not this approximation is a good one is revealed by the usefulness of the final series in representing the required solution of the equations of motion.

The trial solution (6) is, in fact, an exact solution of the approximation $e^Q L_2[\phi] = 0$, with the function $A(\xi)$ undefined. ψ_0 does not, of course, satisfy the full equations and a correction term is added:

$$\psi = [A\phi_0 + \epsilon\phi_1]\,e^Q. \tag{7}$$

Since the approximation used in deriving (6) neglects terms of order $R^{-\frac{1}{2}}$ it may be anticipated that ϵ will be equal to this small parameter, but this will emerge naturally in the iteration process.

It is desired to solve

$$L_1[\psi] = 0.$$

From (7) this is

$$L_1[A\phi_0 e^Q] = -\epsilon L_1[e^Q\phi_1]. \tag{8}$$

On replacing the right-hand side by the approximate form $e^Q L_2[\phi_1]$, and noting that $L_2[\phi_0] = 0$, it is clear that the left-hand side reduces to the difference between the exact and the approximate forms of the operator acting on ψ_0. These difference terms are $O(R^{-\frac{1}{2}})$ and putting $\epsilon = R^{-\frac{1}{2}}$ (8) reduces to

$$AF_0 + \frac{dA}{d\xi}F_1 + \frac{d^2A}{d\xi^2}F_2 + \frac{d^3A}{d\xi^3}F_3 + \frac{d^4A}{d\xi^4}F_4 = L_2[\phi_1(\xi,\eta)]. \tag{9}$$

The coefficients F_2, F_3 and F_4 are small (of order R^{-1} or smaller) compared with F_0 and F_1, and the initial trial solution is therefore modified by an amplitude function dominated by the first two terms. Use of the adjoint function $\Phi(\xi,\eta)$ enables the solubility condition

$$\int_0^\infty \Phi L_2[\phi]\,d\eta = 0$$

to be applied (see Stuart 1960). The amplitude function is given by the ordinary differential equation

$$A(\xi)\int_0^\infty F_0\,\Phi\,d\eta+\frac{dA}{d\xi}\int_0^\infty F_1\,\Phi\,d\eta+\text{etc.} = 0, \tag{10}$$

or

$$AG_0+\frac{dA}{d\xi}G_1+\frac{d^2A}{d\xi^2}G_2+\frac{d^3A}{d\xi^3}G_3+\frac{d^4A}{d\xi^4}G_4 = 0, \tag{11}$$

where

$$G_j = \int_0^\infty F_j\,\Phi\,d\eta \quad \text{for} \quad j = 0,1,2,3,4.$$

G_2, G_3 and G_4 are small and the significant slow root can be extracted. The iterative process can only be expected to yield a useful result when the development of the mean flow is slow and the correction to the zero-order parallel flow solution is small. It was assumed at the outset that the amplitude function $A(\xi)$ was a slow function of ξ, and roots involving large derivatives of $A(\xi)$ are thus not consistent with this assumption and do not relate to valid solutions of the problem under consideration. At any station ξ, the left-hand side of (9) is known and ϕ_1 can, in principle at least, be evaluated from the inhomogeneous Orr–Sommerfeld equation. Further terms in the series can then be found by evaluating successive correction terms in descending powers of $R^{\frac{1}{2}}$. In many practical examples R will be sufficiently large for the behaviour of the disturbance to be described adequately by the leading term, namely $A(\xi)\,\phi_0 e^Q$. In this case there is no virtue in evaluating F_0, F_1, etc. further than $O(R^{-\frac{1}{2}})$, and we find on neglecting F_2, F_3 and F_4 that (11) reduces to

$$A^{-1}\,dA/d\xi = -\hat{G}_0/\hat{G}_1, \tag{12}$$

where the caret implies evaluation to order $R^{-\frac{1}{2}}$.

The first correction to the parallel flow solution $A(\xi)$ can be obtained for all $\xi > 1$ from (12).

3. Comparison between theory and experiment

In parallel flows the eigenfunction $\phi(\eta)$ is independent of the streamwise station ξ, and the exponential part of the perturbation stream function uniquely defines the wavelength and amplification rate. The behaviour of any unsteady physical quantity along any path $\eta =$ constant is governed by this exponent. The controlling parameters for non-parallel flow are, however, not uniquely defined in this way and rather greater care is required when comparing theory with experiment, even in weakly non-parallel flow situations. The eigenfunction changes slowly with the streamwise station and estimates of the wavelength and amplification therefore depend, to a small extent, on how these quantities are defined.

Experimental studies of instability waves rely almost exclusively on measurements made with hot-wire anemometers. Simple hot-wire elements detect the streamwise component u of the fluctuations. The behaviour of u as a function of streamwise station is generally used to define stability. Measured values of the exponent depend on the height of the probe above the surface. In addition, different estimates are obtained by traversing the probe downstream either at a constant physical distance y from the boundary, or at a constant non-dimensional distance η. Schubauer chose to measure amplification with a probe at a fixed distance from the plate at some position below the point of maximum velocity fluctuations. He suggested that amplification rates measured there would be reduced slightly from the 'true' value by the effect of the boundary-layer growth and the associated outward movement of the velocity maximum.

Ross *et al.* (1970) were also well aware of these difficulties in making meaningful comparisons between theoretical and experimental amplification rates. Their measurements were taken with the hot wire at a constant non-dimensional distance from the surface and corrections based on theoretical eigenfunctions were applied to compensate for errors introduced through variations in the eigenfunction shape with Reynolds number.

The various methods of defining the growth rate lead to results differing by amounts of order $R^{-\frac{1}{2}}$. Since the parallel flow approximation neglects terms of this order it would be inconsistent to attempt to improve on the simple measure α for calculating the amplification rate and wavenumber. Although agreement between theory and experiment may be apparently improved by making a particular choice of probe height and method of correction for changes in the eigenfunction, this is a highly illusory result. The agreement between theory based on the Orr–Sommerfeld model and experiment cannot be better than $O(R^{-\frac{1}{2}})$. It is, however, essential to take due account of the above factors if full advantage is to be realized from any higher-order theoretical treatment such as that presented in this paper.

The perturbation stream function arises as a series in $R^{-\frac{1}{2}}$:

$$\psi = \{A(\xi)\phi_0(\xi,\eta) + R^{-\frac{1}{2}}\phi_1(\xi,\eta) + O(R^{-1})\}\, e^{Q}, \tag{13}$$

$$u = \frac{\partial \psi}{\partial y} = \frac{U}{\nu R^{\frac{1}{2}}}\left\{\frac{A\phi_0'}{\xi^{\frac{1}{2}}} + \frac{\phi_1'}{\xi^{\frac{1}{2}}R^{\frac{1}{2}}} + O(R^{-1})\right\} e^{Q},$$

where primes denote differentiation with respect to η. The relative rate of change of the u component is

$$\frac{1}{u}\frac{\partial u}{\partial \xi} = \frac{i\alpha R^{\frac{1}{2}}}{\xi^{\frac{1}{2}}} + \left\{\frac{1}{A}\frac{dA}{d\xi} + \frac{1}{\phi_0'}\frac{\partial \phi_0'}{\partial \xi} - \frac{1}{2\xi}\right\} + O(R^{-\frac{1}{2}}). \tag{14}$$

The leading term is the wavenumber, which arises solely from the Orr–Sommerfeld approximation, while the terms in the group next in order of importance arise through the amplitude function, the eigenvalue modification with Reynolds number and the co-ordinate system respectively. The imaginary part of (14) gives the wavenumber correct to $O(R^{-\frac{1}{2}})$, and the real part gives the amplification rate.

The neutral curve is defined by

$$\xi^{-\frac{1}{2}}\alpha_i - R^{-\frac{1}{2}}\left\{\left(\frac{1}{A}\frac{dA}{d\xi}\right)_r + \left(\frac{1}{\phi_0'}\frac{\partial\phi_0'}{\partial\xi}\right)_r - \frac{1}{2\xi}\right\} = 0 \tag{15}$$

for the $|u|$ component of velocity. Similar relations can be generated for other quantities such as $|v|$ or the local kinetic energy $\overline{u^2}+\overline{v^2}$, which has been used by Bouthier. For any particular experiment the appropriate neutral curve can be calculated from some relationship like (15) and valid comparisons made. It would be convenient to use some integral parameter, such as kinetic energy for example, to characterize the stability and thus remove any ambiguity in the definition of the neutral curve. Although the kinetic-energy integral is a physically meaningful parameter, it is a quantity that is difficult to measure. A more useful parameter for defining amplification is the integral of $\overline{u^2}$ across the flow. Other possible definitions are given by the values of the growth of $(\overline{u^2})^{\frac{1}{2}}$ at either of the two maxima in η.

Bouthier chose to define the amplification rate using the growth of the local kinetic energy $\overline{u^2}+\overline{v^2}$. He obtained the lowest Reynolds number where the amplification first reached zero and an upper limit where the growth was positive at every point through the boundary layer.

Neutral loops are obtained here for various parameters using the first-order correction terms to solutions of the Orr–Sommerfeld equation.

The quantities chosen to define amplification are as follows.

(a) The kinetic-energy integral:

$$E = \int_0^\infty (\overline{u^2}+\overline{v^2})\,dy,$$

$$\alpha^{(a)} = \frac{\xi}{E}\frac{dE}{d\xi} = 2\left\{-\xi^{\frac{1}{2}}R^{\frac{1}{2}}\alpha_i + \left[\left(\frac{\xi}{A}\frac{dA}{d\xi}\right)_r + \left(\frac{\xi}{e}\frac{de}{d\xi}\right)_r - \frac{1}{4}\right]\right\},$$

where

$$e = \int_0^\infty [\phi_0'\tilde{\phi}_0' + \alpha\tilde{\alpha}\phi_0\tilde{\phi}_0]\,d\eta.$$

(b) The integral of $\overline{u^2}$:

$$I = \int \overline{u}^2\,d\eta,$$

$$\alpha^{(b)} = \frac{\xi}{I}\frac{dI}{d\xi} = 2\left\{-\xi^{\frac{1}{2}}R^{\frac{1}{2}}\alpha_i + \left[\left(\frac{\xi}{A}\frac{dA}{d\xi}\right)_r + \left(\frac{\xi}{h}\frac{dh}{d\xi}\right)_r - \frac{1}{2}\right]\right\},$$

where

$$h = \int_0^\infty [\phi_0'\tilde{\phi}_0']\,d\eta.$$

(c) The u component:

$$\alpha^{(c)} = \frac{\xi}{|u|}\frac{d|u|}{d\xi} = -\xi^{\frac{1}{2}}R^{\frac{1}{2}}\alpha_i + \left[\left(\frac{\xi}{A}\frac{dA}{d\xi}\right)_r + \left(\frac{\xi}{\phi_0'}\frac{\partial\phi_0'}{\partial\xi}\right)_r - \frac{1}{2}\right],$$

where $(\xi/\phi_0')\,\partial\phi_0'/\partial\xi$ is evaluated at (i) the point where $|u|$ is a maximum, (ii) the point where $|u|$ is a minimum, (iii) the station $\eta = 0{\cdot}08$, which is approximately $y/\delta = 0{\cdot}15$, the position used by Ross *et al.* for their experiments, and (iv) fixed values of y below the point of maximum u for comparison with measurements of Schubauer & Skramstad.

4. Computation

It was necessary to solve the Orr–Sommerfeld equation to obtain the required information for calculating the functions G_0 and G_1. A shooting technique was used with a Runge–Kutta integration combined with Kaplan filtering to remove the spurious divergent mode from the solution. The computation was carried out on a KDF9 which holds real numbers to 48 bits (11 decimal digits). The integration range was split into two: 20 steps were used in the range $0 < \eta < 2$ and 40 steps for $2 < \eta < 8$. The eigenvalues obtained agreed with those given by Jordinson (1970) to $O(10^{-5})$. The eigenfunction and its derivatives as well as the adjoint functions were calculated and stored on magnetic tape for a range of Reynolds numbers and non-dimensional frequencies ($F = \beta/R_\delta$). This information was then used to evaluate $\hat{G}_0$ and $\hat{G}_1$, and hence the various amplification contours.

5. Results

The various amplification rates defined in § 3 were obtained numerically using (12) and (A 6) for a range of Reynolds numbers and frequency parameters. These were used to find the neutral boundaries which separate the stable from the unstable domain in the F, R plane. Figure 1 compares the neutral curves based on integral quantities $\alpha^{(a)}$ and $\alpha^{(b)}$. The loop based on energy flux lies outside that given by $\alpha_i = 0$ and furthermore values of the amplification rate within the unstable zone exceed the parallel flow values. For the energy flux, therefore, the growth of the boundary layer leads to a reduction of stability. This arises partly because the energy integral contains the boundary-layer thickness, which increases like $x^{\frac{1}{2}}$. The neutral curve based on the integral of the u component squared, which does not contain this factor, shows quite close agreement with the Orr–Sommerfeld neutral loop.

These integral parameters are not the most convenient measures of amplification as far as the experimenter is concerned; simpler direct estimates are given by the u component of the fluctuations at selected stations in the boundary layer. The root-mean-square value of u has two maxima, one large peak close to the wall and a weaker second peak at the outer edge of the boundary layer. The large inner maximum has been suggested as suitable for amplification measurements (Ross *et al.* 1970), but the outer one should also be considered as a possible station since this maximum is somewhat flatter and its position does not have to be defined quite so precisely. Neutral loops based on these two particular stations are shown on figure 2. Schubauer & Skramstad made their measurements at some unspecified position below the inner maximum and these are plotted on figure 3 with the theoretical neutral curve calculated for a position halfway between the wall and the peak. Figure 4 shows the experimental data given by Ross *et al.* (1970) together with the appropriate theoretical curve for $y/\delta = 0{\cdot}15$.

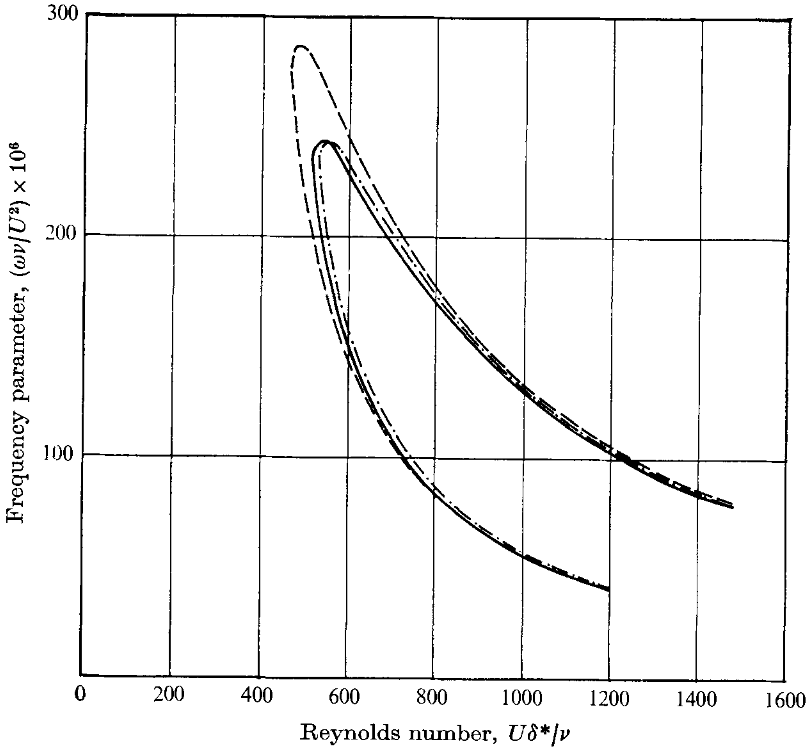

FIGURE 1. Neutral amplification curves based on integral parameters. ——, $\alpha_i = 0$, for parallel flow; – – –, $\alpha^{(a)} = 0$, the kinetic energy; —·—, $\alpha^{(b)} = 0$, the integral of $\overline{u}^2$.

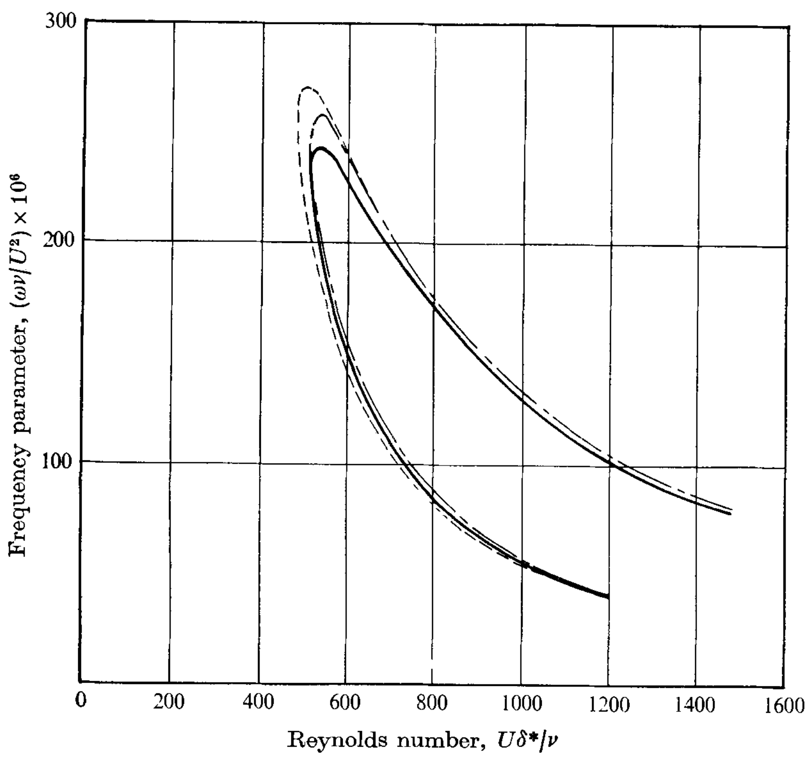

FIGURE 2. Neutral loops based on the points of maximum $|u|$. ——, $\alpha_i = 0$, for parallel flow; – – –, $\alpha^{(c)} = 0$, for inner maximum; —–—, $\alpha^{(c)} = 0$, for outer peak.

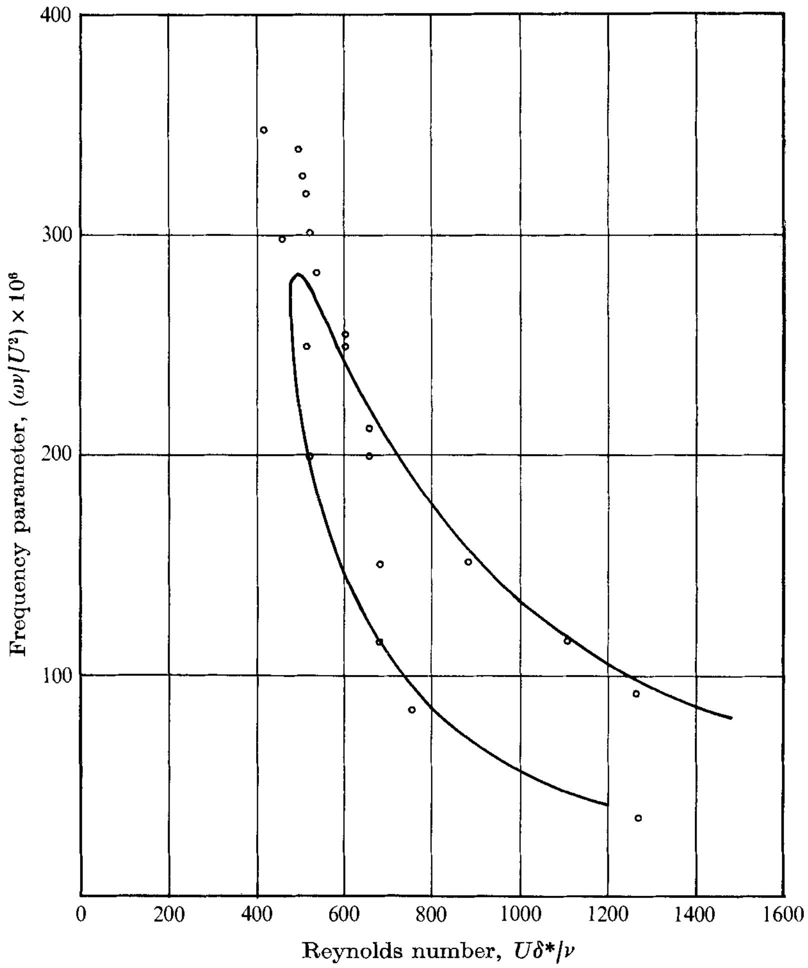

FIGURE 3. Comparison with data of Schubauer & Skramstad.

6. Discussion

The Orr–Sommerfeld parallel flow solution has been used as a zero-order approximation to the full equations of motion and a correction in the form of an amplitude scaling function has been found together with an inhomogeneous equation for the first correction to the eigenfunction. This equation has not been solved and thus the overall motion has only been obtained to zero order. Nevertheless, the amplification rate and wavelength, which only require knowledge of the disturbance to lowest order, are given to order $R^{-\frac{1}{2}}$. The equation defining the amplitude function to this order is identical to that given by Bouthier. The way the eigenfunctions are normalized does not influence the solution since the amplitude equation effectively contains this information.

The figures showing the stability boundaries for different parameters clearly demonstrate the need for care when comparing observations with theoretical predictions. The various amplification rates differ from the parallel flow value

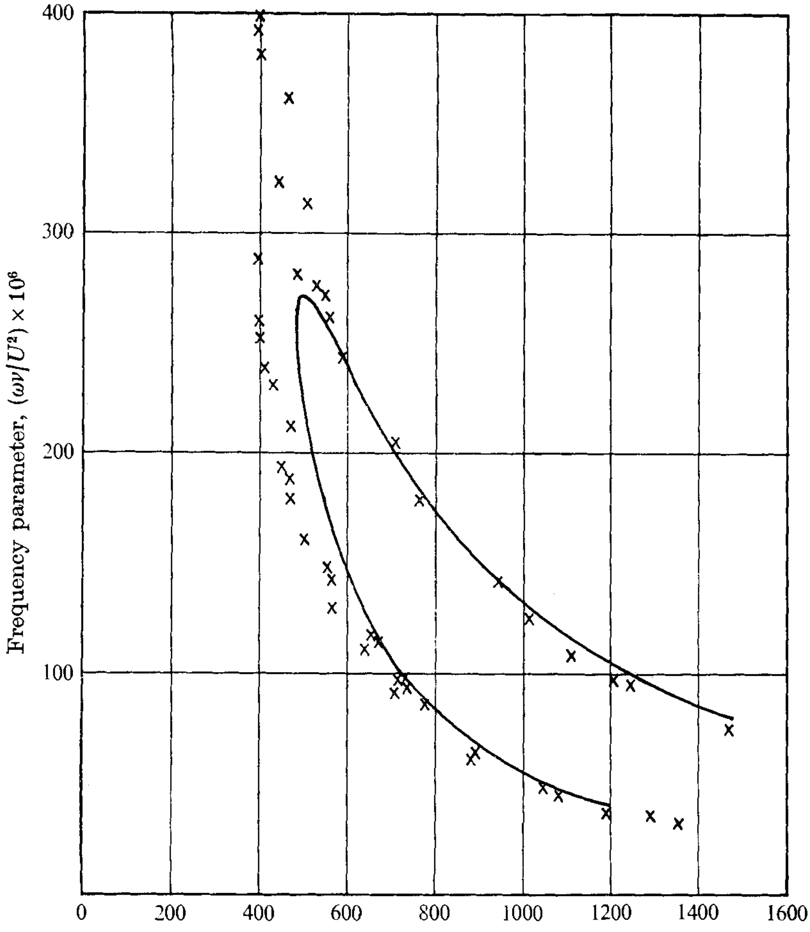

FIGURE 4. Comparison of theory with data of Ross *et al.*

by amounts of order $R^{-\frac{1}{2}}$ and this is significant near the critical Reynolds number. Figures 3 and 4 compare experimental data with the present first-order theory for the u component at positions appropriate to the experiments. In both examples agreement between experiment and theory has been improved by the first-order correction to the amplification, but the high frequency/low Reynolds number points are not explained. Bouthier compared the same data with his theoretical neutral curve based on the local energy flux and achieved remarkable agreement. It is not at all obvious why this correlation is so strong. Ling & Reynolds found almost no influence of boundary-layer growth on the neutral curve, but it is not clear to what parameter the calculated amplification factors refer. It appears that the corrections to the wavenumber that were calculated refer to the contribution from the amplitude equation, but this quantity depends on the normalization used and by itself is insufficient to define the behaviour of the instability to the necessary order.

Both the method of direct expansion and of successive approximation can be continued to yield higher terms of the series expansion for the perturbation stream function. If the mean flow is assumed to be given solely by the boundary-layer approximation (only one term in the series for $\bar{\psi}$) Bouthier's multiple-scale expansion appears to be particularly direct, successive terms of the series being generated by the same basic ordinary differential operator. Since the solution at any streamwise location is controlled by local values of the coefficients of this operator, the behaviour is parabolic, like the zero-order term arising from the Orr–Sommerfeld equation. Although the full equations of motion are elliptic, it is interesting to note that the series solution retains this parabolic character to arbitrary order. Of course the mean flow cannot be defined solely by the boundary-layer equations and the above expansion procedure can only be extended by one further term before other mean flow terms arise in the equations and destroy the simplicity of the process. Nevertheless, a study of the first few terms of such a series for the disturbance stream function, generated in a purely 'boundary-layer' mean flow, would be useful in establishing the behaviour of the series, and might offer some guidance concerning the accuracy of any resulting partial sum.

Extension of the iterative method to higher order, even in the somewhat artificial case of a 'boundary-layer' mean flow, leads to solutions with rather different characteristics. The amplitude equation (11) contains some algebraically weak terms of second and higher order. These may be neglected up to the second iteration, but they have to be taken into account when higher terms are being evaluated. The resulting higher-order differential equation for the amplitude function requires additional boundary conditions to define the constants of integration. These derivatives arise from the biharmonic part of the Navier–Stokes equations and are connected with its elliptic character. The iterative series solution exhibits parabolic behaviour up to second order, after which elliptic characteristics become evident. It seems likely that the indeterminacy of the higher-order terms of the series arises from some inadequacy in posing the problem. If the evaluation of the perturbation was being attempted over the whole field, by say a finite-difference scheme, it would be necessary to specify sufficient boundary conditions around the perimeter of the domain, and in the situation being discussed here values downstream would be required. This information is not available and the elliptic nature of the full equations prevents a solution being obtained to higher than second order by the iterative procedure. Since the method of multiple scales can apparently overcome this difficulty, it may in some way filter out that part of the solution which is purely parabolic, but it is not certain whether this result is physically meaningful. In the case of the stability of the flow over a flat plate sufficiently accurate solutions are given by the first two terms of the series and in practice the difficulties discussed above do not arise. In the case of a finite flat plate the mean flow stream function contains terms of order one, which are identically zero for the semi-infinite plate, and the functions $\hat{G}_0$ and $\hat{G}_1$ will be modified. No assessment has been made of these effects, but it is felt that in practice a long enough plate will produce only a very weak deviation from boundary-layer behaviour and intuitively it is believed that the result given in appendix A will be adequate to define the solution.

It may be anticipated that the solution generated by the iterative scheme would consist of a series of terms such that the nth term is of order ϵ^n, where ϵ is some suitable small parameter defining any departure of the mean flow from a two-dimensional structure. The terms can thus be expected initially to decrease in magnitude, although at some stage the coefficients may become so large that the series diverges. An approximate solution is obtained by truncating the series at the smallest term, the error being related to the first term neglected. In the boundary-layer problem the small parameter is $R^{-\frac{1}{2}}$. Since this quantity has a numerical value of around 300 near the critical Reynolds number, we can expect the solution obtained here to be sufficiently accurate. In more general flows where it is not possible to generate analytical expressions for the mean flow it probably will be impossible to define a simple small parameter. However, if the mean flow is defined, say numerically, it is quite feasible to generate a series by the iteration process. The numerical values of the terms of this series will offer some guide as to whether or not the series represents a useful solution. It seems likely that moderate deviations of the mean flow can be adequately treated and that the first correction to the parallel flow solution would often suffice. In situations of rapid mean flow distortion it can be expected that the iteratively generated series will diverge and the 'short wavelength' approximation will no longer be appropriate. A 'long wavelength' treatment must be sought, where the behaviour of the disturbance is controlled by some integrated property of the mean flow and is not directly influenced by rapid local distortions.

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Appendix A

The linear Navier–Stokes operator for a mean flow stream function $\overline{\psi}$ is

$$L_1[\;] = \frac{\partial\nabla^2}{\partial t} + \frac{\partial\overline{\psi}}{\partial y}\frac{\partial\nabla^2}{\partial x} + \frac{\partial\nabla^2\overline{\psi}}{\partial x}\frac{\partial}{\partial y} - \frac{\partial\overline{\psi}}{\partial x}\frac{\partial\nabla^2}{\partial y} - \frac{\partial\nabla^2\overline{\psi}}{\partial y}\frac{\partial}{\partial x} - \nu\nabla^4. \tag{A 1}$$

For the semi-infinite flat plate the stream function of the mean flow can be obtained as a series in the form

$$\overline{\psi} = \nu\xi^{\frac{1}{2}}R^{\frac{1}{2}}f(\eta) + 0\times O(1) + O(R^{-\frac{1}{2}}), \tag{A 2}$$

where $f(\eta)$ is the 'boundary-layer' solution given by the Blasius equation

$$f''' + \tfrac{1}{2}ff'' = 0. \tag{A 3}$$

The result of operating on $A\phi(\xi,\eta)\,e^Q$ and (A 2) by (A 1) can be written in the form

$$Ae^QL_2[\phi] + \theta \tag{A 4}$$

in terms of the co-ordinates η and ξ, using the relations

$$\frac{\partial}{\partial x} = \frac{U}{\nu R}\left\{\frac{\partial}{\partial\xi} - \frac{\eta}{2\xi}\frac{\partial}{\partial\eta}\right\}, \quad \frac{\partial}{\partial y} = \frac{U}{\nu\xi^{\frac{1}{2}}R^{\frac{1}{2}}}\frac{\partial}{\partial\eta},$$

$$L_2[\phi] = (iU^4/\nu^3R^{\frac{3}{2}}\rho^{\frac{3}{2}})\{(\alpha f' - \beta)(\phi'' - \alpha^2\phi) - \alpha f'''\phi - i\phi^{\mathrm{iv}}/R^{\frac{1}{2}}\xi^{\frac{1}{2}}\},$$

and θ, a function completely specified in terms of f, ϕ, G and A, contains terms with coefficients $R^{-\frac{1}{2}}$, R^{-1}, etc.

The fourth-order viscous term must be included if solutions generated by equating $L_2[\phi]$ to zero are to be valid approximations for the full equations of motion and the four hydrodynamic boundary conditions. These solutions are highly oscillatory in some regions of the flow and differentiation with respect to η increases the relative importance of a term there. Thus even though the viscous term contains a factor $R^{-\frac{1}{2}}$, the four differentiations with respect to η make this term dominant close to the boundary and near the 'critical-layer' singularity, which occurs when $\alpha f' = \beta$. The ordering of terms, taking the vertical structure into account in the above way, is discussed in appendix B. For the purpose of generating iterative solutions it is sufficient that θ, the terms neglected in forming the first approximation, be significantly smaller than those retained and it is immaterial whether or not other small terms are also included in L_2. It is, of course, essential to include only terms which maintain L_2 in an integrable form. The complete viscous term $\nu\nabla^4\psi$ is generally incorporated in the first-order approximation: the Orr–Sommerfeld equation

$$(\alpha f' - \beta)(\phi'' - \alpha^2\phi) - \alpha f'''\phi = (i/R^{\frac{1}{2}}\xi^{\frac{1}{2}})\{\phi^{\text{iv}} - 2\alpha^2\phi'' + \alpha^4\phi\}. \qquad \text{(A 5)}$$

The above form was used in the present work. Barry & Ross (1970) included some additional terms, which arise from the vertical component of the mean flow.

To proceed with the iteration it is necessary to evaluate the quantities F_0 and F_1 from solutions of (A 5). These functions can be written out in full in terms of f, ϕ_0 and Q, and evaluated exactly. However, since it is only intended here to evaluate G_0 and G_1 to first order, some terms may be neglected. The magnitudes of the various terms can be assessed in the sense discussed in appendix B. To order $R^{-\frac{1}{2}}$

$$\hat{F}_0\xi = [2\alpha\beta - 3\alpha^2 f' - f'''] \left[\xi\frac{\partial\phi_0}{\partial\xi} - \frac{\eta}{2}\phi_0'\right] + f'\left\{\xi\frac{\partial\phi_0''}{\partial\xi} - \frac{\eta}{2}\phi_0''' - \phi_0''\right\}$$
$$+ (\beta - 3\alpha f')\left(\xi\frac{d\alpha}{d\xi} - \frac{\alpha}{2}\right)\phi_0 - \tfrac{1}{2}\{(f'' + \eta f''')\phi_0' + (f - \eta f')(\phi_0''' - \alpha^2\phi_0')\}$$

and

$$\hat{F}_1 = [2\alpha\beta - 3\alpha^2 f' - f''']\phi_0 + f'\phi_0''. \qquad \text{(A 6)}$$

Appendix B

The relative magnitudes of the various terms arising in the expansion for θ can be estimated from known properties of the fundamental asymptotic solutions of the Orr–Sommerfeld equation (see Lin 1955, p. 34). The equation has two slow roots described by the inviscid second-order Rayleigh equation and two rapidly oscillating viscous modes which behave like $\exp[\pm R^{\frac{1}{4}}g(\eta)]$. In the boundary-layer case the outer boundary condition requires that the divergent viscous mode be discarded. The remaining viscous mode together with the inviscid solution must satisfy the wall boundary conditions of zero normal and tangential velocity. This requires the ratio of amplitudes to be $O(R^{-\frac{1}{4}}):O(1)$ for the viscous and inviscid stream functions. Differentiation of the viscous solution in the normal direction n times raises its order by $R^{\frac{1}{4}n}$ and the overall order of magnitude becomes $R^{\frac{1}{3}(n-1)}$, compared with the inviscid part of the solution. The complete perturbation stream function is composed of both viscous and inviscid modes. The order of magnitude of a term, ϕ^n say, is $O(1)$ for an inviscid mode and $O(R^{\frac{1}{3}(n-1)})$ for a viscous one.

What is really required here is an estimate of the contribution of any term in the integrals G_j, so that only the relevant parts be retained. Since the viscous modes are only large in layers of thickness $O(R^{-\frac{1}{3}})$ it is appropriate to define the magnitude of the contribution from a viscous term as $O(R^{\frac{1}{3}(n-2)})$. The viscous component dominates when $n > 2$. Thus if $n \leqslant 2$ the importance of ϕ^n is $O(1)$, but if $n > 2$ it is considered to be $O(R^{\frac{1}{3}(n-2)})$. Using this criterion it can be shown that all the terms retained in (A 6) are greater than $O(R^{-\frac{1}{3}})$.

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